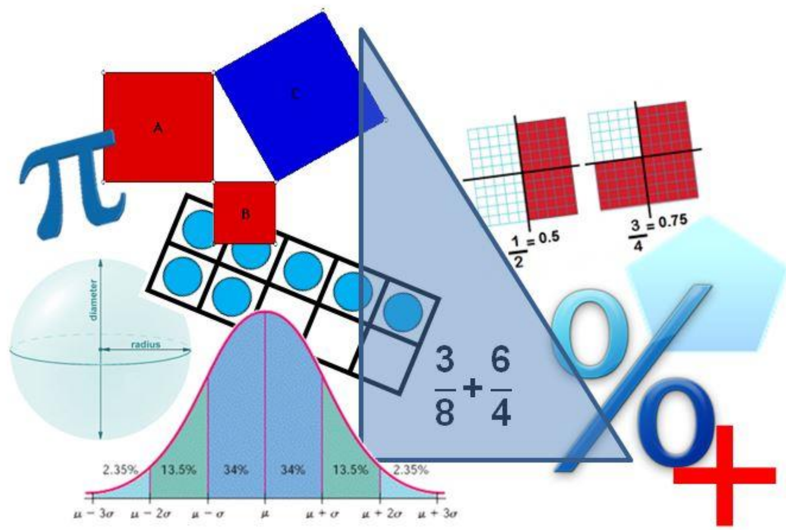
Mathematics

2016 Standards of Learning

Discrete Mathematics
Curriculum Framework



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Virginia 2016 *Mathematics Standards of Learning Curriculum Framework*
Introduction

The 2016 *Mathematics Standards of Learning Curriculum Framework*, a companion document to the 2016 *Mathematics Standards of Learning*, amplifies the *Mathematics Standards of Learning* and further defines the content knowledge, skills, and understandings that are measured by the Standards of Learning assessments. The standards and *Curriculum Framework* are not intended to encompass the entire curriculum for a given grade level or course. School divisions are encouraged to incorporate the standards and *Curriculum Framework* into a broader, locally designed curriculum. The *Curriculum Framework* delineates in greater specificity the minimum content that all teachers should teach and all students should learn. Teachers are encouraged to go beyond the standards as well as to select instructional strategies and assessment methods appropriate for all students.

The *Curriculum Framework* also serves as a guide for Standards of Learning assessment development. Students are expected to continue to connect and apply knowledge and skills from Standards of Learning presented in previous grades as they deepen their mathematical understanding. Assessment items may not and should not be a verbatim reflection of the information presented in the *Curriculum Framework*.

Each topic in the 2016 *Mathematics Standards of Learning Curriculum Framework* is developed around the Standards of Learning. The format of the *Curriculum Framework* facilitates teacher planning by identifying the key concepts, knowledge, and skills that should be the focus of instruction for each standard. The *Curriculum Framework* is divided into two columns: Understanding the Standard and Essential Knowledge and Skills. The purpose of each column is explained below.

Understanding the Standard
This section includes mathematical content and key concepts that assist teachers in planning standards-focused instruction. The statements may provide definitions, explanations, examples, and information regarding connections within and between grade level(s)/course(s).

Essential Knowledge and Skills
This section provides a detailed expansion of the mathematics knowledge and skills that each student should know and be able to demonstrate. This is not meant to be an exhaustive list of student expectations.

Mathematical Process Goals for Students

The content of the mathematics standards is intended to support the following five process goals for students: becoming mathematical problem solvers, communicating mathematically, reasoning mathematically, making mathematical connections, and using mathematical representations to model and interpret practical situations. Practical situations include real-world problems and problems that model real-world situations.

Mathematical Problem Solving

Students will apply mathematical concepts and skills and the relationships among them to solve problem situations of varying complexities. Students also will recognize and create problems from real-world data and situations within and outside mathematics and then apply appropriate strategies to determine acceptable solutions. To accomplish this goal, students will need to develop a repertoire of skills and strategies for solving a variety of problems. A major goal of the mathematics program is to help students apply mathematics concepts and skills to become mathematical problem solvers.

Mathematical Communication

Students will communicate thinking and reasoning using the language of mathematics, including specialized vocabulary and symbolic notation, to express mathematical ideas with precision. Representing, discussing, justifying, conjecturing, reading, writing, presenting, and listening to mathematics will help students clarify their thinking and deepen their understanding of the mathematics being studied. Mathematical communication becomes visible where learning involves participation in mathematical discussions.

Mathematical Reasoning

Students will recognize reasoning and proof as fundamental aspects of mathematics. Students will learn and apply inductive and deductive reasoning skills to make, test, and evaluate mathematical statements and to justify steps in mathematical procedures. Students will use logical reasoning to analyze an argument and to determine whether conclusions are valid. In addition, students will use number sense to apply proportional and spatial reasoning and to reason from a variety of representations.

Mathematical Connections

Students will build upon prior knowledge to relate concepts and procedures from different topics within mathematics and see mathematics as an integrated field of study. Through the practical application of content and process skills, students will make connections among different areas of mathematics and between mathematics and other disciplines, and to real-world contexts. Science and mathematics teachers and curriculum writers are encouraged to develop mathematics and science curricula that support, apply, and reinforce each other.

Mathematical Representations

Students will represent and describe mathematical ideas, generalizations, and relationships using a variety of methods. Students will understand that representations of mathematical ideas are an essential part of learning, doing, and communicating mathematics. Students should make connections among different representations – physical, visual, symbolic, verbal, and contextual – and recognize that representation is both a process and a product.

Instructional Technology

The use of appropriate technology and the interpretation of the results from applying technology tools must be an integral part of teaching, learning, and assessment. However, facility in the use of technology shall not be regarded as a substitute for a student's understanding of quantitative and algebraic concepts and relationships or for proficiency in basic computations. Students must learn to use a variety of methods and tools to compute, including paper and pencil, mental arithmetic, estimation, and calculators. In addition, graphing utilities, spreadsheets, calculators, dynamic applications, and other technological tools are now standard for mathematical problem solving and application in science, engineering, business and industry, government, and practical affairs.

Calculators and graphing utilities should be used by students for exploring and visualizing number patterns and mathematical relationships, facilitating reasoning and problem solving, and verifying solutions. However, according to the National Council of Teachers of Mathematics, "… the use of calculators does not supplant the need for students to develop proficiency with efficient, accurate methods of mental and pencil-and-paper calculation and in making reasonable estimations." State and local assessments may restrict the use of calculators in measuring specific student objectives that focus on number sense and computation. On the grade three state assessment, all objectives are assessed without the use of a calculator. On the state assessments for grades four through seven, objectives that are assessed without the use of a calculator are indicated with an asterisk (*).

Computational Fluency

Mathematics instruction must develop students' conceptual understanding, computational fluency, and problem-solving skills. The development of related conceptual understanding and computational skills should be balanced and intertwined, each supporting the other and reinforcing learning.

Computational fluency refers to having flexible, efficient and accurate methods for computing. Students exhibit computational fluency when they demonstrate strategic thinking and flexibility in the computational methods they choose, understand and can explain, and produce accurate answers efficiently.

The computational methods used by a student should be based on the mathematical ideas that the student understands, including the structure of the base-ten number system, number relationships, meaning of operations, and properties. Computational fluency with whole numbers is a goal of mathematics instruction in the elementary grades. Students should be fluent with the basic number combinations for addition and subtraction to 20 by the end of grade two and those for multiplication and division by the end of grade four. Students should be encouraged to use computational methods and tools that are appropriate for the context and purpose.

Algebra Readiness

The successful mastery of Algebra I is widely considered to be the gatekeeper to success in the study of upper-level mathematics. "Algebra readiness" describes the mastery of, and the ability to apply, the *Mathematics Standards of Learning*, including the Mathematical Process Goals for Students, for kindergarten through grade eight. The study of algebraic thinking begins in kindergarten and is progressively formalized prior to the study of the algebraic content found in the Algebra I Standards of Learning. Included in the progression of algebraic content is patterning, generalization of arithmetic concepts, proportional reasoning, and representing mathematical relationships using tables, symbols, and graphs. The K-8 *Mathematics Standards of Learning* form a progression of content knowledge and develop the reasoning necessary to be well-prepared for mathematics courses beyond Algebra I, including Geometry and Statistics.

Equity

> "Addressing equity and access includes both ensuring that all students attain mathematics proficiency and increasing the numbers of students from all racial, ethnic, linguistic, gender, and socioeconomic groups who attain the highest levels of mathematics achievement."
>
> – National Council of Teachers of Mathematics

Mathematics programs should have an expectation of equity by providing all students access to quality mathematics instruction and offerings that are responsive to and respectful of students' prior experiences, talents, interests, and cultural perspectives. Successful mathematics programs challenge students to maximize their academic potential and provide consistent monitoring, support, and encouragement to ensure success for all. Individual students should be encouraged to choose mathematical programs of study that challenge, enhance, and extend their mathematical knowledge and future opportunities.

Student engagement is an essential component of equity in mathematics teaching and learning. Mathematics instructional strategies that require students to think critically, to reason, to develop problem-solving strategies, to communicate mathematically, and to use multiple representations engages students both mentally and physically. Student engagement increases with mathematical tasks that employ the use of relevant, applied contexts and provide an appropriate level of cognitive challenge. All students, including students with disabilities, gifted learners, and English language learners deserve high-quality mathematics instruction that addresses individual learning needs, maximizing the opportunity to learn.

DM.1[†] **The student will model problems, using vertex-edge graphs. The concepts of valence, connectedness, paths, planarity, and directed graphs will be investigated.**

Understanding the Standard	Essential Knowledge and Skills
• A tournament is a digraph that results from giving directions to the edges of a complete graph. • Adjacent vertices are connected by an edge. • In a connected graph, every pair of vertices is adjacent. • Graphs can be used to solve problems including food chains and number of paths.	**The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to** • Determine the valence of each vertex in a graph. • Use graphs to model situations in which the vertices represent objects, and edges (drawn between vertices) represent a particular relationship between objects. • Represent the vertices and edges of a graph as an adjacency matrix, and use the matrix to solve problems. • Investigate and describe valence and connectedness. • Determine whether a graph is planar or nonplanar. • Use directed graphs (digraphs) to represent situations with restrictions in traversal possibilities.

[†] Standard should be included in a one-semester course in Discrete Mathematics.

DM.2[†] **The student will solve problems through investigation and application of circuits, cycles, Euler paths, Euler circuits, Hamilton paths, and Hamilton circuits. Optimal solutions will be sought using existing algorithms and student-created algorithms.**

Understanding the Standard	Essential Knowledge and Skills
• If *G* is a connected graph and all its valences are even, then *G* has an Euler circuit. • Pairs of routes (circuits) correspond to the same Hamilton circuit because one route can be obtained from the other by traversing the vertices in reverse order. There are $\frac{(n-1)!}{2}$ Hamilton circuits. • A multigraph is connected if there is a path between every pair of vertices.	**The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to** • Determine whether a graph has an Euler circuit or path, and determine it, if it exists. • Determine whether a graph has a Hamilton circuit or path, and determine it, if it exists. • Count the number of Hamilton circuits for a complete graph with *n* vertices. • Use an Euler circuit algorithm to solve optimization problems.

[†] Standard should be included in a one-semester course in Discrete Mathematics.

DM.3[†] The student will apply graphs to conflict-resolution problems, such as map coloring, scheduling, matching, and optimization.

Understanding the Standard	Essential Knowledge and Skills
• Every planar graph has a chromatic number that is less than or equal to four (the four-color-map theorem). • A graph can be colored with two colors if and only if it contains no cycle of odd length. • The chromatic number of a graph cannot exceed one more than the maximum number of degrees of the vertices of the graph.	**The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to** • Model projects consisting of several subtasks, using a graph. • Use graphs to resolve conflicts that arise in scheduling. • Determine the chromatic number of a graph.

[†] Standard should be included in a one-semester course in Discrete Mathematics.

DM.4 The student will apply algorithms relating to trees, networks, and paths. Appropriate technology will be used to determine the number of possible solutions and generate solutions when a feasible number exists.

Understanding the Standard	Essential Knowledge and Skills
• A spanning tree of a connected graph *G* is a tree that is a subgraph of *G* and contains every vertex of *G*.	**The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to** • Use Kruskal's algorithm to determine the shortest spanning tree of a connected graph. • Use Prim's algorithm to determine the shortest spanning tree of a connected graph. • Use Dijkstra's algorithm to determine the shortest spanning tree of a connected graph.

DM.5† **The student will analyze and describe the issue of fair division in discrete and continuous cases.**

Understanding the Standard	Essential Knowledge and Skills
• Group decision making combines the wishes of many to yield a single fair result. • A fair division problem may be discrete or continuous. • The success of the estate division algorithm requires that each heir be capable of placing a value on each object in the estate. • A fair division problem consists of *n* individuals (players) who must partition some set of goods, *s*, into *n* disjoint sets.	**The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to** • Investigate and describe situations involving discrete division (e.g., estate division). • Use an algorithm for fair division for a group of indivisible objects. • Investigate and describe situations involving continuous division of an infinitely divisible set (e.g., cake cutting). • Use an algorithm for fair division of an infinitely divisible set.

† Standard should be included in a one-semester course in Discrete Mathematics.

DM.6† The student will investigate and describe weighted voting and the results of various election methods. These may include approval and preference voting as well as plurality, majority, runoff, sequential runoff, Borda count, and Condorcet winners.

Understanding the Standard	Essential Knowledge and Skills
• Historically, popular voting methods have often led to counterintuitive results. • A candidate who wins over every other candidate in a one-on-one ballot is a Condorcet winner. • A Borda count assigns points in descending order to each voter's subsequent ranking and then adds these points to arrive at a group's final ranking. • To select a voting system is to compromise between the shortcomings inherent in each system.	**The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to** • Determine in how many different ways a voter can rank choices. • Investigate and describe the following voting procedures: – weighted voting; – plurality; – majority; – sequential (winners runoff); – sequential (losers are eliminated); – Borda count; and – Condorcet winner. • Compare and contrast different voting procedures. • Describe the possible effects of approval voting, insincere and sincere voting, a preference schedule, and strategic voting on the election outcome.

† Standard should be included in a one-semester course in Discrete Mathematics.

DM.7 The student will identify apportionment inconsistencies that apply to issues such as salary caps in sports and allocation of representatives to Congress. Historical and current methods will be compared.

Understanding the Standard	Essential Knowledge and Skills
• The apportionment of congressional representatives is based on the latest census.	**The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to** • Compare and contrast the Hamilton and Jefferson methods of political apportionment with the Hill-Huntington method (currently in use in the U.S. House of Representatives) and the Webster-Willcox method. • Solve allocation problems, using apportionment methods. • Investigate and describe how salary caps affect apportionment.

DM.8 The student will describe and apply sorting algorithms and coding algorithms used in sorting, processing, and communicating information.

Understanding the Standard	Essential Knowledge and Skills
• A bubble sort orders elements of an array by comparing adjacent elements. • A merge sort combines two sorted lists into a single sorted list. • Coding algorithms must account for the number of possible codes within the constraints of the coding system.	**The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to** • Select and apply a sorting algorithm, such as a – bubble sort; – merge sort; and – network sort. • Describe and apply a coding algorithm, such as – ISBN numbers; – UPC codes; – Zip codes; and – banking codes.

DM.9[†] **The student will select, justify, and apply an appropriate technique to solve a logic problem.**

Understanding the Standard	Essential Knowledge and Skills
• Two-valued (Boolean) algebra serves as a workable method for interpreting the logical truth and falsity of compound statements. • Venn diagrams provide pictures of topics in set theory, such as intersection and union, mutually exclusive sets, and the empty set.	**The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to** • Generate truth tables that encode the truth and falsity of two or more statements. • Use Venn diagrams to represent set relationships, such as intersection and union. • Interpret Venn diagrams. • Use Venn diagrams to codify and solve logic problems. • Use matrices as arrays of data to solve logic problems.

[†] Standard should be included in a one-semester course in Discrete Mathematics.

DM.10 The student will use algorithms to schedule tasks in order to determine a minimum project time. The algorithms will include critical path analysis, the list-processing algorithm, and student-created algorithms.

Understanding the Standard	Essential Knowledge and Skills
• Critical path scheduling sometimes yields optimal solutions.	**The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to** • Specify in a digraph the order in which tests are to be performed. • Identify the critical path to determine the earliest completion time (minimum project time). • Use the list-processing algorithm to determine an optimal schedule. • Create and test scheduling algorithms.

DM.11 The student will solve linear programming problems.

Understanding the Standard	Essential Knowledge and Skills
• Linear programming models an optimization process. • A linear programming model consists of a system of constraints and an objective quantity that can be maximized or minimized. • Any maximum or minimum value for a system of inequalities will occur at a corner point of a feasible region.	**The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to** • Model practical problems with systems of linear inequalities. • Identify the feasibility region of a system of linear inequalities with no more than four constraints. • Identify the coordinates of the corner points of a feasibility region. • Determine the maximum or minimum value of the system. • Describe the meaning of the maximum or minimum value in terms of the original problem.

DM.12 The student will use the recursive process and difference equations with the aid of appropriate technology to generate
a) compound interest;
b) sequences and series;
c) fractals;
d) population growth models; and
e) the Fibonacci sequence.

Understanding the Standard	Essential Knowledge and Skills
• Recursion is a process that creates new objects from existing objects that were created by the same process. • A fractal is a figure whose dimension is not a whole number. • Fractals are self-similar.	**The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to** • Use finite differences and recursion to model compound interest and population growth situations. • Model arithmetic and geometric sequences and series recursively. • Compare and contrast the recursive process, and create fractals. • Compare and contrast the recursive process and the Fibonacci sequence. • Determine a recursive relationship that generates the Fibonacci sequence.

DM.13 The student will apply the formulas of combinatorics in the areas of
a) the Fundamental (Basic) Counting Principle;
b) knapsack and bin-packing problems;
c) permutations and combinations; and
d) the pigeonhole principle.

Understanding the Standard	Essential Knowledge and Skills
• Combinatorics is the branch of mathematics that addresses the number of ways objects can be arranged or combined. • If n and r are positive integers and $n \geq r$, $n\,P\,r = \frac{n!}{(n-r)!}$ and $n\,C\,r = \frac{n!}{r!\,(n-r)!}$. • A bin-packing problem determines the minimum number of containers of fixed volume (bins) required to hold a set of objects. • A knapsack problem determines the most valuable set of objects that fit into a container (knapsack) of fixed volume. • Bin packing and knapsack packing are optimization techniques.	**The student will use problem solving, mathematical communication, mathematical reasoning, connections, and representations to** • Determine the number of combinations possible when subsets of r elements are selected from a set of n elements without regard to order. • Use the Fundamental (Basic) Counting Principle to determine the number of possible outcomes of an event. • Use the knapsack and bin-packing algorithms to solve practical problems. • Determine the number of permutations possible when r objects selected from n objects are ordered. • Use the pigeonhole principle to solve packing problems to facilitate proofs.